Differential Equation

Definition

A differential equation is an eqⁿ involving independent variables, dependent variables and different derivatives of the dep. variables w.r.t independent variables.

Exp If x is an independent variable, y is a dependent variable, then any eqⁿ involving x, y and $\frac{dy}{dx}$ is called a differential Eqⁿ.

Exp (1) $\frac{dy}{dx} + x^2 = 2$

(2) $\frac{dy}{dx} + 3x = 0$

(3) $\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$

(4) $\left(\frac{dy}{dx}\right)^2 + 3y^2 = 4x$

All the above differential eqⁿs are called Ordinary differential eqⁿs due to the presence of only one independent variable.

Partial Differential Equation

If the number of independent variables is more than one, then the derivatives are partial and the differential equation is called a partial differential equation.

Exp. (1) $x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = nu$.

(2) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Order of a Differential Eqn

The order of a differential eqn is the highest order of the derivative occurring in it.

Exp. (1) $\frac{dy}{dx} + x^2 = 1$, order = 1

(2) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 1 = 0$, order = 2

(3) $\frac{d^3y}{dx^3} = 0$, order = 3

(4) $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$, order = 2

Degree of a differential eqn

The degree of a diff. eqn is the highest exponent of the highest order derivative after the eqn has been free from radicals and fractions.

for ex ① $\left(\frac{dy}{dx}\right)^2 + 3y^2 = 5x$ degree = 2

② $\frac{d^2y}{dx^2} = \sqrt{3 + \frac{dy}{dx}}$

By removing radicals,

$\left(\frac{d^2y}{dx^2}\right)^2 = 3 + \frac{dy}{dx}$, so, degree = 2

③ $\frac{dy}{dx} = \frac{3}{dy/dx}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 3$, so, degree = 2.

Linear and Non-Linear differential Eqn

A diff. eqn is said to be Linear if it satisfies the following conditions—

① every dep. variable and its derivative are of 1st degree

② Dep. variable and its derivative are not multiplied together.

Otherwise it is called Non-Linear diff. eqn.

Example

① $\frac{dy}{dx} = x$

② $\frac{d^2y}{dx^2} = \sin x$

③ $\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

} Linear.

Non-Linear

Ex! ① $\left(\frac{dy}{dx}\right)^2 = x+y$

② $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$

} non-Linear.

Example Find the order and degree of the following diff. eq's –

① $\frac{dy}{dx} + xy = 1$

② $\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

Soln ① Given eqn is $\frac{dy}{dx} + xy = 1$

order = 1

degree = 1

② $\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

$\Rightarrow \sqrt{1-y^2} + y\sqrt{1-x^2} \cdot \frac{dy}{dx} = 0$

Here order = 1, since the highest order derivative is 1st order.

degree = 1.

Solution of the differential eqn of 1st order and 1st degree

① If the eqn is of the form

$\frac{dy}{dx} = f(x)$ then the solution is obtained by direct integration only

Ex ① Solve $\frac{dy}{dx} = x^2 + \sin 3x$.

Soln Given $\frac{dy}{dx} = x^2 + \sin 3x$

$\Rightarrow dy = (x^2 + \sin 3x)\, dx$

$\Rightarrow \int dy = \int (x^2 + \sin 3x)\, dx$

$\Rightarrow \boxed{y = \frac{x^3}{3} - \frac{\cos 3x}{3} + c}$

② ~~Solve~~ If the eqn is of the form

$\frac{dy}{dx} = f(y)$ then

$\frac{dy}{f(y)} = dx$ ~~and~~ which on integrations gives the solution

Ex Solve $\frac{dy}{dx} = y + 2$

Soln $\frac{dy}{dx} = y + 2$

$\Rightarrow \frac{dy}{y+2} = dx$

$\Rightarrow \int \frac{dy}{y+2} = \int dx$

$\Rightarrow \log(y+2) = x + c$

$\Rightarrow \boxed{y + 2 = e^{x+c}}$

(3) Equation with variable separation form —

ie if the eqⁿ is in the form

$f(x)\,dx + f(y)\,dy = 0$, then the solution is obtained by integrating each term separately.

for ex (1) $\frac{dy}{dx} = (e^x+1)y$

Solⁿ $\frac{dy}{dx} = (e^x+1)y$

$\Rightarrow \frac{dy}{y} = (e^x+1)\,dx$

which on integration gives

$\log y = e^x + x + c$

(2) Solve $x(1+y^2)\,dx + y(1+x^2)\,dy = 0$

Solⁿ Given eqⁿ is $x(1+y^2)\,dx + y(1+x^2)\,dy = 0$

$\Rightarrow \frac{x(1+y^2)\,dx + y(1+x^2)\,dy}{(1+x^2)(1+y^2)} = 0$

$\Rightarrow \int \frac{x}{1+x^2}\,dx + \int \frac{y}{1+y^2}\,dy = 0$

$\Rightarrow \frac{1}{2}\int \frac{2x}{1+x^2}\,dx + \frac{1}{2}\int \frac{2y}{1+y^2}\,dy = 0$

$\Rightarrow \frac{1}{2}\log(1+x^2) + \frac{1}{2}\log(1+y^2) = C$

$\Rightarrow \log(1+x^2) + \log(1+y^2) = 2c$

$\Rightarrow \log[(1+x^2)\cdot(1+y^2)] = \log k$ (Let $2c = \log k$)

$\Rightarrow \boxed{(1+x^2)(1+y^2) = k}$

is the required solution.

Solving Differential Eq^s of 2nd order which is of the form $\frac{d^2y}{dx^2} = f(x)$

In this case since the eq^n is of 2nd order, so, we'll integrate 2 times and get two consts of integration.

Example Solve $\frac{d^2y}{dx^2} = 6x + \sec^2 x$

Sol^n Given $\frac{d^2y}{dx^2} = 6x + \sec^2 x$

which on integration gives

$$\frac{dy}{dx} = \int (6x + \sec^2 x)\, dx$$

$$= 6\frac{x^2}{2} + \tan x + c_1$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 3x^2 + \tan x + c_1}$$ where c_1 is the const.

Again integrating both sides w.r.t. x

$$\int \frac{dy}{dx} \cdot dx = \int (3x^2 + \tan x + c_1)\, dx$$

$$\Rightarrow y = 3 \cdot \frac{x^3}{3} + \log \sec x + c_1 x + c_2$$

$$\Rightarrow \boxed{y = x^3 + \log(\sec x) + c_1 x + c_2}$$

which is the required solution to the given differential equation.

~~Only~~ Solve the following problems:

(1) $\frac{dy}{dx} = e^{x+y} + x^2 e^y$

(2) $(1+x)(1+y^2)\,dx + (1+y)(1+x^2)\,dy = 0$

(3) $\frac{d^2y}{dx^2} = \cos x - \sin x$

(4) Find the order and degree of the given differential equations

(a) $\left(\frac{dy}{dx}\right)^3 + y^3 = \frac{d^2y}{dx^2}$

(b) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = 3\frac{d^2y}{dx^2}$

(5) Solve $\frac{dy}{dx} = x\log x$.

(6) $\frac{d^2y}{dx^2} = x$

(7) $\frac{d^3y}{dx^3} = 5$

(8) $\frac{dy}{dx} = \sin(x+y)$

(9) $\frac{dy}{dx} + 1 = e^{x+y}$

(10) $\frac{d^2y}{dx^2} = e^x + \cos x$.

Homogeneous Equations

Homogeneous ~~Equation~~ function :-

A function $f(x,y)$ is called a homogeneous fun. of degree n if the degree of each term is n.

Ex ① $f(x,y) = x^2+y^2-xy$

is a homogeneous fun of degree 2 since each term is of degree 2.

② $f(x,y) = x^2-xy^2+x^2y$

is not a homogeneous fun, because each term are not of same degree.

Homogeneous Differential Equation

A diff. eqn $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ is called a homo. diff. eqn if both $f(x,y)$ and $g(x,y)$ are homogeneous functions of same degree in x and y.

Ex $\frac{dy}{dx} = \frac{x^2-y^2}{xy}$

is a homo. diff. eqn of degree 2

Method of Solving a homogeneous differential Equation :—

Given $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ —— (1) is a homogeneous differential eqn

Step-1 put $y = vx$ so that

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Step-2 Substituting the above values, eqn (1) reduces to

$$v + x\frac{dv}{dx} = f(v)$$

$$\Rightarrow x\frac{dv}{dx} = f(v) - v$$

$$\Rightarrow \frac{dv}{f(v) - v} = \frac{dx}{x}$$

which on integration gives the solution.

Step-3 ~~Replace~~ put $\boxed{v = \frac{y}{x}}$ to get the required solution to the given diff. eqn.

Example 1 Solve $\frac{dy}{dx} = \frac{y-x}{x+y}$.

Sol^n Given $\frac{dy}{dx} = \frac{y-x}{x+y}$ ——— (1)

which a homogeneous fn^ of degree 1.

since each terms are of 1st degree.

Putting $y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ ∴ eq^ (1)

reduces to

$$v + x\frac{dv}{dx} = \frac{vx - x}{x + vx}$$

$$= \frac{\cancel{x}(v-1)}{\cancel{x}(1+v)}$$

$\Rightarrow$ ~~$v - \frac{v-1}{v+1}$~~

$$\Rightarrow x\frac{dv}{dx} = \frac{v-1}{v+1} - v$$

$$= \frac{v-1-v(v+1)}{v+1}$$

$$= \frac{\cancel{v}-1-v^2-\cancel{v}}{v+1} = \frac{-(1+v^2)}{1+v}$$

$$\Rightarrow x\frac{dv}{dx} = -\left(\frac{1+v^2}{1+v}\right)$$

$$\Rightarrow \frac{dv}{-\left(\frac{1+v^2}{1+v}\right)} = \frac{dx}{x}$$

$$\Rightarrow \frac{dx}{x} = -\left(\frac{1+v}{1+v^2}\right) dv$$

$$\Rightarrow -\frac{dx}{x} = \frac{1}{1+v^2}dv + \frac{v}{1+v^2}dv$$

which on integration gives

$$\int -\frac{dx}{x} = \int \frac{dv}{1+v^2} + \int \frac{v\,dv}{1+v^2}$$

$$\Rightarrow -\log x = \tan^{-1} v + \frac{1}{2}\log(1+v^2) + c$$

$$\Rightarrow \tan^{-1} v + \frac{1}{2}\log(1+v^2) + \log x = c$$

$$\Rightarrow \tan^{-1} v + \log\sqrt{1+v^2} + \log x = c$$

$$\Rightarrow \tan^{-1} v + \log\left(x\sqrt{1+v^2}\right) = c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} + \log\left(x \cdot \sqrt{1+\frac{y^2}{x^2}}\right) = c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} + \log\left(x \cdot \sqrt{\frac{x^2+y^2}{x^2}}\right) = c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} + \log\sqrt{x^2+y^2} = c$$

$$\Rightarrow \tan^{-1}\frac{y}{x} + \log(x^2+y^2)^{1/2} = c$$

$$\Rightarrow \boxed{\tan^{-1}\frac{y}{x} + \frac{1}{2}\log(x^2+y^2) = c}$$

is the sol^n to the given diff. eq^n.

Try to Solve the differential equations

(1) $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$.

(2) $x^2\, dy + y(x+y)\, dx = 0$

using homogeneous method.

Equations reducible to Homogeneous Form :—

Equation of the type $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$

can be reduced to homogeneous form by the substitution

$$\left.\begin{aligned} x &= X+h \\ y &= Y+K \end{aligned}\right\}$$ where h, K are constants whose values are to be determined.

so that $\frac{dy}{dx} = \frac{dY}{dX}$

Then the eq^n becomes homogeneous and can be solved by substituting $Y = vX$

~~and can be~~ using Homogeneous Method.

Example Solve $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

Sol: Putting $x = X+h$ and $y = Y+k$ we have

$$\frac{dY}{dX} = \frac{X+h+2(Y+k)-3}{2(X+h)+Y+k-3}$$

$$= \frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)} \quad \text{--- (1)}$$

Taking $h+2k-3=0$ --- (2)

and $2h+k-3=0$ --- (3)

$2 \times$ Eq^n (2) $\Rightarrow$ $2h+4k-6=0$

$1 \times$ Eq^n (3) $\Rightarrow$ $2h+k-3=0$

$\Rightarrow 3k-3=0 \Rightarrow 3k=3 \Rightarrow \boxed{k=1}$

Putting $k=1$ in eq^n (2)

$h+2-3=0 \Rightarrow \boxed{\cancel{h=3}} \Rightarrow \boxed{h=1}$

Now putting h and k in eq^n (1)

$$\frac{dY}{dX} = \frac{X+2Y}{2X+Y} \quad \text{--- (4)}$$

which is homogeneous.

Putting $Y = vX$ so that $\frac{dY}{dX} = v + X\frac{dv}{dX}$ in eq^n (1)

$$v + X\frac{dv}{dX} = \frac{X+2vX}{2X+vX}$$

$$= \frac{X(1+2v)}{X(2+v)}$$

$$\Rightarrow v + X\frac{dv}{dX} = \frac{1+2v}{2+v}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{1+2v}{2+v} - v = \frac{1+2v - v(2+v)}{2+v}$$

$$= \frac{1+2v-2v-v^2}{2+v}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{1-v^2}{2+v}$$

$$\Rightarrow \frac{2+v}{1-v^2}\,dv = \frac{dX}{X}$$ which on integration

gives $$\int \frac{2+v}{1-v^2}\,dv = \int \frac{dX}{X}$$

$$\Rightarrow \int \frac{2}{1-v^2}\,dv + \int \frac{v}{1-v^2}\,dv = \int \frac{dX}{X}$$

$$\Rightarrow 2\log\frac{1+v}{1-v} - \frac{1}{2}\log(1-v^2) = \log X + \log C$$

$$\Rightarrow \log\frac{1+v}{1-v} - \log\sqrt{1-v^2} = \log CX$$

$$\Rightarrow \log\left(\frac{\frac{1+v}{1-v}}{\sqrt{1-v^2}}\right) = \log CX$$

$$\Rightarrow CX = \frac{1+v}{1-v}\cdot\frac{1}{\sqrt{1-v^2}}$$

$$= \frac{1+v}{(1-v)^{3/2}}$$

$$\Rightarrow C^2X^2(1-v)^3 = 1+v$$

$$\Rightarrow C^2X^2\left(1-\frac{Y}{X}\right)^3 = 1+\frac{Y}{X}$$

$$\Rightarrow C^2(X-Y)^3 = X+Y$$

$$\Rightarrow \boxed{c^2(x-y)^3 = x+y-2}$$ (putting $X = x-1$, $Y = y-1$)

which is the required solution.

Exact Equation

Consider the differential equations

$$x\,dy + y\,dx = 0$$

and $$\frac{y\,dx - x\,dy}{y^2} = 0$$

~~which can be written~~ The LHS of the eqns can be written as $d(xy)$ and $d\left(\frac{x}{y}\right)$ respectively.

Exact Differential Equation

The differential eqn

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{——— (1)}$$

is called exact if $\exists$ (there exist) a function $u(x,y)$ s.t

$$d(u(x,y)) = M(x,y)\,dx + N(x,y)\,dy.$$

Test for Exactness

The diff. eqn $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact iff $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$.

Ex1 Test whether the diff. eqn

$$e^y\,dx + (xe^y + 2y)\,dy = 0$$

is exact or not.

Soln Given diff. eqn is

$$e^y\,dx + (xe^y + 2y)\,dy = 0$$

where $M(x,y) = e^y$, $N(x,y) = xe^y + 2y$

Now $\frac{\partial M}{\partial y} = \frac{\partial (e^y)}{\partial y} = e^y$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(xe^y + 2y)$$

$$= \frac{\partial}{\partial x}(xe^y) + \frac{\partial}{\partial x}(2y)$$

$$= e^y \cdot 1 + 0 = e^y$$

Now $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow$ The given eqn is exact.

If the eqn $M\,dx + N\,dy = 0$ —— (1) is not exact then ~~by multiplying a factor~~ it is made exact by multiplying a factor called Integrating factor and then can be solved by Exact Method.

Integrating factor (I.F)

I.F is a factor which when multipling with ~~on~~ a non-exact diff eqn it becomes exact.

Note The number of integrating factor is infite.

Rules for finding I.F

1. If $Mx + Ny \neq 0$ and eq^n 1 is homogeneous then $\frac{1}{Mx + Ny}$ is an I.F

2. If $Mx - Ny \neq 0$ and eq^n 1 is of the form $f_1(x,y) \cdot y\,dx + f_2(x,y)\,x\,dy = 0$ then $\frac{1}{Mx - Ny}$ is an I.F

3. If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = f(y)$, a fu^n of y only then $\boxed{I.F = e^{\int -f(y)\,dy}}$

4. If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, a fu^n of x only then $\boxed{I.F = e^{\int f(x)\,dx}}$

Linear Differential Equation

Def^n :- A differential eq^n in which the dependent variable and all its derivatives occur in the first degree only and are not multiplied together, is called a Linear Differential equation.

Example (1)

$$\frac{dx}{dy} + 5y = x^2$$

It is a linear eq^n of order 1

(2) $y \frac{dy}{dx} + 5y = x$

It is a non-Linear eq^n since y and its derivatives $\frac{dy}{dx}$ are multiplied together.

General form of Linear Eq^n

The differential equation of the form

$$\boxed{\frac{dy}{dx} + P(x) \cdot y = Q(x)}$$

is called a Linear eq^n where $P(x)$ and $Q(x)$ are functions of x only.

Here $\boxed{I.F = e^{\int P(x)\, dx}}$

After finding the I.F, the general solution is given by

$$\boxed{y \cdot I.F = \int [Q(x) \cdot I.F]\, dx}$$

Example

Solve $x \frac{dy}{dx} + 3y = x^2$

Sol^n The given eq^n is $x \frac{dy}{dx} + 3y = x^2$

~~which is Linear of the form~~ $\Rightarrow \frac{dy}{dx} + \frac{3}{x} \cdot y = x$

which is Linear

ie of the form $\frac{dy}{dx} + P(x) \cdot y = Q(x)$

where $P(x) = \frac{3}{x}$

$Q(x) = x$

$\therefore$ I.F $= e^{\int P\,dx} = e^{\int \frac{3}{x}\,dx}$

$= e^{3\log x} = e^{\log x^3}$

$= x^3$

$\Rightarrow \boxed{\text{I.F} = x^3}$

So, the general solution is given by

$y \cdot \text{IF} = \int (Q(x) \cdot \text{IF})\,dx$

$\Rightarrow y \cdot x^3 = \int x \cdot x^3\,dx$

$= \int x^4\,dx = \frac{x^5}{5} + C$

$\Rightarrow \boxed{y x^3 = \frac{x^5}{5} + C}$

Ans

Solve the following differential equations

(1) $4\frac{dy}{dx} + 8y = 5e^{-3x}$

(2) $\frac{dy}{dx} + (\sec x)\cdot y = \tan x$

(3) $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

(4) $\frac{dy}{dx} + y\cot x = \cos x$

(5) $x\frac{dy}{dx} + 2y = 4x^2$

~~This is all about Diff eqn~~